POLAR CREMONA TRANSFORMATIONS AND MONODROMY OF POLYNOMIALS

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ABSTRACT. Consider the gradient map associated to any non-constant homogeneous polynomial $f \in \mathbb{C}[x_0, \ldots, x_n]$ of degree d, defined by

$$\phi_f = grad(f) \colon D(f) \to \mathbb{P}^n, (x_0 \colon \ldots \colon x_n) \to (f_0(x) \colon \ldots \colon f_n(x))$$

where $D(f) = \{x \in \mathbb{P}^n; f(x) \neq 0\}$ is the principal open set associated to f and $f_i = \frac{\partial f}{\partial x_i}$. This map corresponds to polar Cremona transformations. In Proposition 3.4 we give a new lower bound for the degree $d(f)$ of ϕ_f under the assumption that the projective hypersurface $V : f = 0$ has only isolated singularities. When $d(f) = 1$, Theorem 4.2 yields very strong conditions on the singularities of V.

1. INTRODUCTION

Consider the gradient map associated to any non-constant homogeneous polynomial $f \in \mathbb{C}[x_0, \ldots, x_n]$ of degree d, defined by

$$\phi_f = grad(f) \colon D(f) \to \mathbb{P}^n, (x_0 \colon \ldots \colon x_n) \to (f_0(x) \colon \ldots \colon f_n(x))$$

where $D(f) = \{x \in \mathbb{P}^n; f(x) \neq 0\}$ is the principal open set associated to f and $f_i = \frac{\partial f}{\partial x_i}$. This map corresponds to the polar Cremona transformations considered by Dolgachev in [10], see also [9], [7], [2], [12].

In section 2 we recall basic facts on the degree $d(f)$ of the gradient map $grad(f)$, emphasizing in Theorem 2.3 the relation to the Bouquet Theorem of Lê in [14].

In section 3 we consider the challenging Conjecture 3.1 due to A. Dimca and S. Papadima in [9], saying roughly that $d(f) > 1$ for most projective hypersurface $V : f = 0$ having only isolated singularities. In Proposition 3.4 we find a new lower bound for the degree of the gradient map ϕ_f in this case.

In the last section we show how the monodromy of a polynomial function $h : \mathbb{C}^n \to \mathbb{C}$, naturally associated to f, gives extremely strong conditions on the monodromy operators associated to the singularities of V if $d(f) = 1$.

2. The Degree of The Gradient

Let $d(f) = deg(\phi_f)$ denote the degree of the gradient map, which is defined as follows. For a dominant map ϕ_f we have the following equivalent definitions:

(i) There is a Zariski open and dense subset U in $\mathbb{P}^n$ such that for all $u \in U$ the fiber $\phi_f^{-1}(u)$ has exactly $d(f)$ points;

(ii) the rational fraction field extension $\phi_f^*\colon K(\mathbb{P}^n) \to K(D(f))$ has degree $d(f)$, see Mumford [15], Proposition (3.17).

In particular, this latter formulation implies that $d(f) = 1$ if and only if the gradient map ϕ_f induces a birational isomorphism of the projective space $\mathbb{P}^n$.

The degree of the gradient map ϕ_f is defined to be zero if the gradient map is not dominant.

Note that in all the above we may replace the open set $D(f)$ by the larger open set

$$E(f) = \mathbb{P}^n \backslash \{x \in \mathbb{P}^n;\ f_0(x) = f_1(x) = \ldots = f_n(x) = 0\}$$

without changing the degree of the gradient map (to see this just see description (i) given above for the degree).

One has the following topological description of the degree $d(f)$ of the gradient map $grad(f)$, see [9].

Theorem 2.1. *For any non-constant homogeneous polynomial $f \in \mathbb{C}[x_0, \ldots, x_n]$, the complement $D(f)$ is homotopy equivalent to a CW complex obtained from $D(f) \cap H$ by attaching $d(f)$ cells of dimension n, where H is a generic hyperplane in $\mathbb{P}^n$. In particular, we have*

$$d(f) = (-1)^n \chi(D(f) \backslash H)$$

Note that the meaning of 'generic' here is quite explicit: the hyperplane H has to be transversal to a stratification of the projective hypersurface V.

This yields in particular the following corollary, see [9] and also [12] for a recent, completely different approach.

Corollary 2.2. *The degree of the gradient map $grad(f)$ depends only on the reduced polynomial f_{red} associated to f.*

Moreover, Theorem 2.1 can be restated in the following way, which shows that for any projective hypersurface V, if we choose the hyperplane at infinity H in a generic way, then the topology of the affine part $X = V \backslash H$ is very simple. For details, see [9].

Theorem 2.3. *For any non-constant homogeneous polynomial $f \in \mathbb{C}[x_0, \ldots, x_n]$, the affine part $X(f) = V(f) \backslash H$ of the corresponding projective hypersurface $V(f)$ with respect to a generic choice of the hyperplane at infinity H is homotopy equivalent to a bouquet of $(n-1)$-spheres. The number of spheres in this bouquet is the degree $d(f)$.*

Remark 2.4. (i) Using Thom's second Isotopy Lemma, see for instance [5], it follows that, for any projective variety V, the topology of affine part $X = V \backslash H$ is independent of H, for a generic hyperplane H. For this reason, we will use the alternative simpler notation V_a for the generic affine piece X of the projective variety V. Exactly the same argument as in the proof of Theorem 2.3 shows that V_a is homotopy equivalent to a bouquet of k-spheres when V is a complete intersection of dimension k.
(ii) It is not difficult to construct projective hypersurfaces V with isolated singularities such that for a given hyperplane H_0, $V \backslash H_0$ is smooth and contractible, see [3]. However, since H_0 is not a generic hyperplane in this case, this does not imply $d(f) = 0$.

3. Hypersurfaces with Isolated Singularities

In this section we consider the following conjecture, see [9], end of section 3, and [7].

Conjecture 3.1. *Let $f \in \mathbb{C}[x_0, \ldots, x_n]$ be a reduced homogeneous polynomial such that*
(a) $d = deg(f) > 2$ and $n > 2$;
(b) the associated projective hypersurface $V(f)$ has only isolated singularities.
Then $d(f) \neq 1$.

The following result was obtained by A. Dimca in [7].

Theorem 3.2. *This conjecture is true if either*
(i) all the singularities of the hypersurface $V(f)$ are weighted homogeneous, or
(ii) the hypersurface $V(f)$ is a $\mathbb{Q}$-manifold.

Using Theorem 2.3 and known facts on the topology of special fibers in a deformation of an isolated hypersurface singularity, we have

$$(3.1) \qquad\qquad d(f) = (d-1)^n - \mu(V(f)),$$

where $\mu(V(f))$ is the sum of the Milnor numbers of all the singularities of $V(f)$, see [5], p. 161 for details. When all these singularities are weighted homogeneous, then $\mu(V(f)) = \tau(V(f))$, where $\tau(V(f))$ is the sum of the Tjurina numbers of all the singularities of $V(f)$. The claim (i) above follows from deep results by du Plessis and Wall [11] giving upper bounds for $\tau(V(f))$.

The proof of the second claim in Theorem 3.2 is much easier, and is generalized in the proof of Proposition 3.4 below.

The following example shows that the conjecture is optimal.

Example 3.3. An example with $d(f) = 2$ can be obtained as follows. Let $n = 3$, $d = 3$ and let f be the equation of a cubic surface with singularities $A_1 A_5$ or E_6 (for

the existence of cubic surfaces having these configurations of singularities, see [1]).
Now, using equation (3.1), we get $d(f) = (3-1)^3 - 6 = 2$ in either case.

Proposition 3.4. *Let $f \in \mathbb{C}[x_0, \ldots, x_n]$ be a homogeneous polynomial of degree $d > 2$ and such that $n \geq 3$. If the associated projective hypersurface $V = V(f) \subset \mathbb{P}^n$ has only isolated singularities, say at the points $a_1, \ldots, a_p$, then*

$$d(f) \geq b_{n-2}^0(W_{n-2}^d) - \mu^0(V).$$

Here $b_{n-2}^0(W_{n-2}^d)$ is the primitive middle Betti number of a smooth projective hypersurface W_{n-2}^d of degree d and dimension $n-2$ and

$$\mu^0(V) = \sum_{j=1,p} \mu^0(V, a_j)$$

is the sum of the ranks of the radicals of the Milnor lattices L_j corresponding to the singularities (V, a_j).

Note that the hypersurface $V = V(f)$ is a $\mathbb{Q}$-manifold if and only if $\mu^0(V) = 0$ and that $b_{n-2}^0(W_{n-2}^d) > 1$. In this way Proposition 3.4 implies the second claim in Theorem 3.2.

Proof. Let L be the Milnor lattice of the isolated singularity obtained as the vertex of the affine cone over a generic hyperplane section $W = V \cap H_0$ of the hypersurface V. Consider the lattice morphism

$$\phi_V : \ L_1 \oplus L_2 \oplus \ldots \oplus L_p \xrightarrow{\psi_V} L \xrightarrow{p} \overline{L} = L/\operatorname{Rad} L$$

as in [5] p.161, where ψ_V is the obvious (primitive) embedding of lattices,

$$\operatorname{Rad} L = \{x \in L; \ xy = 0, \text{ for all } y \in L\}$$

and p is the canonical projection. It follows that

(3.2) $$\ker \phi_V = \operatorname{Rad} L \cap (L_1 \oplus L_2 \oplus \ldots \oplus L_p).$$

Next we show that

(3.3) $$\ker \phi_V \subset \operatorname{Rad} L_1 \oplus \operatorname{Rad} L_2 \oplus \ldots \oplus \operatorname{Rad} L_p.$$

Let $v = (v_1, \ldots, v_p) \in \ker \phi_V$. Then, by equation (3.2), $v \in \operatorname{Rad} L$. By definition, this implies that $v.w = 0$ for all $w \in L$. Taking $w = (0, \ldots, 0, b_j, 0, \ldots, 0)$ for any $b_j \in L_j$, we get $v_j b_j = 0$ for all $b_j \in L_j$. It implies that $a_j \in \operatorname{Rad} L_j$ for each $j = 1, \ldots, p$. It follows that (3.3) holds.

Consider the Milnor number of the singularity (V, a_k) given by $\mu(V, a_k) = \operatorname{rank} L_k$ and note that $\mu^0(V)$ denotes the sum $\sum_{k=1,p} \operatorname{rank}(\operatorname{Rad} L_k)$. Now, by the basic

properties of the rank of a $\mathbb{Z}$-linear mapping between free $\mathbb{Z}$-modules of finite type, we have

$$\text{rank Im } \phi_V = \text{rank}(\oplus_{k=1}^p L_k) - \text{rank}(\ker \phi_V) = \sum_{k=1,p} \text{rank }(L_k) - \text{rank }(\ker \phi_V) =$$

$$= \mu(V) - \text{rank }(\ker \phi_V).$$

By equation (3.3), we get $\text{rank }(\ker \phi_V) \leq \mu^0(V)$. Therefore, we get

(3.4) $$\text{rank Im } \phi_V \geq \mu(V) - \mu^0(V) \geq 0.$$

Note that $\text{Im } \phi_V \subset \overline{L}$, which yields

(3.5) $$\text{rank } \overline{L} \geq \text{rank Im } \phi_V \geq \mu(V) - \mu^0(V) \geq 0.$$

On the other hand, we have

$$(d-1)^n = b_{n-1}(W_{n-1}^d) + b_{n-2}(W_{n-2}^d) - 1,$$

see formula (5.3.27) on p.159 in [5]. By the definition of the primitive Betti numbers, this is equivalent to

$$(d-1)^n = b_{n-1}^0(W_{n-1}^d) + b_{n-2}^0(W_{n-2}^d).$$

Using the equations 3.1 and 3.5, it follows that

$$d(f) = (d-1)^n - \mu(V) \geq b_{n-1}^0(W_{n-1}^d) + b_{n-2}^0(W_{n-2}^d) - (rk\,\overline{L} + \mu^0(V)).$$

Moreover, one has $\text{rank } \overline{L} = b_{n-1}^0(W_{n-1}^d)$, as follows from Prop (5.3.24), p.157 in [5], which implies the required result. $\square$

For $n = 3$, W_{n-2}^d is a smooth projective plane curve of degree d. By the degree-genus formula, we have

$$g = \frac{(d-1)(d-2)}{2}.$$

Thus, we get

$$b_1^0(W_1^d) = (d-1)(d-2).$$

This implies the following.

Corollary 3.5. *The Conjecture 3.1 is true for any surface V of degree d such that $\mu^0(V) < (d-1)(d-2) - 1$.*

4. The use of monodromy of polynomial functions

In this section we prove the main result, which is a generalization of Proposition 3.4 above, in the key case $d(f) = 1$. In order to state it we need some preliminaries. Assume as above that the hypersurface $V = V(f) \subset \mathbb{P}^n$ has only isolated singularities, say at the points $a_1, \ldots, a_p$. At each singular point a_j, there is a local Milnor fiber F_j and a monodromy operator $T_j : H_{n-1}(F_j, \mathbb{C}) \to H_{n-1}(F_j, \mathbb{C})$. Let

$$(4.1) \qquad \Delta_j(t) = \det(t \cdot I - T_j)$$

be the corresponding characteristic polynomial and set

$$(4.2) \qquad \Delta_V(t) = \prod_{j=1,p} \Delta_j(t).$$

For any $\lambda \in \mathbb{C}$, we let $\mathrm{mult}_V(\lambda)$ denote the multiplicity of λ as a root of the equation $\Delta_V(t) = 0$. For instance, one clearly has (e.g. by using the proof of Prop. 3.4.7 in [5], p.93)

$$(4.3) \qquad \mathrm{mult}_V(1) = \mu^0(V).$$

Consider the following basic example.

Example 4.1. The homogeneous polynomial $g = x_1^d + \ldots + x_n^d$ has an isolated singularity at the origin $0 \in \mathbb{C}^n$. Then it is well known that the corresponding characteristic polynomial of the monodromy T_0 is given by

$$\Delta_0(t) = \left(\frac{(t^d - 1)^{\chi(G)/d}}{t - 1} \right)^{(-1)^{n-1}}$$

where $G : g(x) = 1$ is the affine Milnor fiber of g, and $\chi(G) = 1 + (-1)^{n-1}\mu(g)$, with $\mu(g) = (d-1)^n$. Note that $\chi(F)/d = \chi(U)$, where $U = \mathbb{P}^{n-1} \setminus V(g)$. Since $V(g)$ is smooth, it follows that $\chi(F)/d = 1 + (-1)^{n-1}b_{n-2}^0(W_{n-2}^d)$.

For any $\lambda \in \mathbb{C}$, we let $\mathrm{mult}_0(\lambda)$ denote the multiplicity of λ as a root of the equation $\Delta_0(t) = 0$. It follows that

$$(4.4) \qquad \mathrm{mult}_0(\lambda) = b_{n-2}^0(W_{n-2}^d) + (-1)^{n-1}$$

when λ is a d-root of unity, $\lambda \neq 1$, and

$$(4.5) \qquad \mathrm{mult}_0(\lambda) = b_{n-2}^0(W_{n-2}^d)$$

when $\lambda = 1$. Now we can state our main result.

Theorem 4.2. *If $d(f) = 1$, then for any d-root of unity λ one has*

$$\mathrm{mult}_V(\lambda) \geq \mathrm{mult}_0(\lambda) - 1.$$

Using the equality (4.3), it follows that, by taking $\lambda = 1$, Theorem 4.2 implies Proposition 3.4 when $d(f) = 1$.

Proof. Assume that the hyperplane at infinity $H_0 : x = 0$ in $\mathbb{P}^n$ is transversal to V. Then, in the affine space $\mathbb{C}^n = \mathbb{P}^n \setminus H_0$, the corresponding affine part V_a is defined by an equation

$$h(x_1, \ldots, x_n) = f(1, x_1, \ldots, x_n) = 0.$$

Since $W = V \cap H_0$ is smooth, it follows that the polynomial h is tame, see [5], p.22. In particular, h has only isolated singularities on $\mathbb{C}^n$ and

$$\sum_x \mu(h, x) = (d-1)^n.$$

If $d(f) = 1$, it follows that the polynomial h has precisely $p+1$ isolated singularities: p of them on the zero fiber $H_1 = V_a = h^{-1}(0)$ and the last one, an A_1 singularity, on a different fiber, say $H_2 = h^{-1}(b)$, for $b \neq 0$.

The generic fiber F_h of h is diffeomorphic to the fiber F in Example 4.1 (where $g = 0$ is an equation for W!), and the monodromy at infinity T_∞ of the polynomial h corresponds, under the identification $E = H_{n-1}(F_h, \mathbb{C}) = H_{n-1}(F, \mathbb{C})$, to the monodromy operator T_0 in Example 4.1.

On the other hand, we have a relation of the type $T_\infty = T_1 \circ T_2$, where T_j denotes the monodromy of h about the singular fiber H_j, for $j = 1, 2$. Let $H = \ker(T_2 - I) \subset E$. It is known that codim $H = 1$, see [6], p.202, Prop. 6.3.19 (iii). The monodromy operator T_∞ is semisimple, so we get a direct sum decomposition

$$E = \oplus_\lambda E_\lambda$$

where E_λ is the λ-eigenspace corresponding to T_∞.

Consider first the case $\lambda \neq 1$. Then one clearly has

$$\dim(E_\lambda \cap H) \geq \dim(E_\lambda) - 1 = \text{mult}_0(\lambda) - 1.$$

On the other hand, for $v \in E_\lambda \cap H$ one has $\lambda v = T_1(v)$, i.e. $E_\lambda \cap H$ is contained in the eigenspace of T_1 corresponding to the eigenvalue λ. It is known that the characteristic polynomial of the monodromy operator T_1 is given by

(4.6) $\det(t \cdot I - T_1) = \Delta_V(t) \cdot (t - 1)$

see [6], p.188, Cor. 6.2.17. This yields the claim in this case, since obviously $\text{mult}_V(\lambda)$ is greater than the dimension of eigenspace of T_1 corresponding to the eigenvalue λ.

In the case $\lambda = 1$, it follows from [8], Theorem 3.1, (ii), that both restrictions $T_1|E_1$ and $T_2|E_1$ are the identity of E_1. In particular, in this case we have

$$\dim(E_1 \cap H) = \dim(E_1) = \text{mult}_0(1).$$

Due to the factor $(t - 1)$ in the formula (4.6), this is exactly what is needed to get the claim in this case.

$\square$

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